A NOTE ON IMMERSION INTERTWINES OF INFINITE GRAPHS

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Abstract. We present a construction of two infinite graphs G_1 and G_2, and of an infinite set $\mathscr{F}$ of graphs such that $\mathscr{F}$ is an antichain with respect to the immersion relation and, for each graph G in $\mathscr{F}$, both G_1 and G_2 are subgraphs of G, but no graph properly immersed in G admits an immersion of G_1 and of G_2. This shows that the class of infinite graphs ordered by the immersion relation does not have the finite intertwine property.

1. Introduction

A *graph* G is a pair $(V(G), E(G))$ where $V(G)$, the set of vertices, is an arbitrary and possibly infinite set, and $E(G)$, the set of edges, is a subset of the set of two-element subsets of $V(G)$. In particular, this definition implies that all graphs in this paper are simple, that is, with no loops or multiple edges. The class of finite graphs will be denoted $\mathscr{G}_{<\infty}$ and the class of graphs whose vertex set is infinite will be denoted by $\mathscr{G}_{\infty}$.

Let G and H be graphs, and let $\mathscr{P}(G)$ denote the set of all nontrivial, finite paths of G. We say H is *immersed* in G if there is a map $\varphi : V(H) \cup E(H) \to V(G) \cup \mathscr{P}(G)$, sometimes abbreviated as $\varphi : H \to G$, such that:

(1) if $v \in V(H)$, then $\varphi(v) \in V(G)$;
(2) if v and v' are distinct vertices of H, then $\varphi(v) \neq \varphi(v')$;
(3) if $e = \{v, v'\} \in E(H)$, then $\varphi(e) \in \mathscr{P}(G)$ and the path $\varphi(e)$ connects $\varphi(v)$ with $\varphi(v')$;
(4) if e and e' are distinct edges of H, then the paths $\varphi(e)$ and $\varphi(e')$ are edge-disjoint; and
(5) if $e = \{v, v'\} \in E(H)$ and v'' is a vertex of H other than v and v', then $\varphi(v'') \notin V(\varphi(e))$.

We call φ an *immersion* and write $H \leq_{\mathrm{im}} G$. It is easy to prove (see [3]) that the relation $\leq_{\mathrm{im}}$ is transitive. If C is a subgraph of H, then the restriction of φ to $V(C) \cup E(C)$ will be abbreviated by $\varphi|_C$. If $\varphi|_{V(H)}$ is a bijection such that two vertices, v and v', of H are adjacent if and only if their images, $\varphi(v)$ and $\varphi(v')$, are adjacent in G, then we say that φ induces an isomorphism between H and G; otherwise φ is *proper*. If $H = G$, then φ is a *self-immersion*, and, if additionally, it induces the identity map, then it is *trivial*. It is worth noting that immersion, as defined above, is sometimes called *strong immersion*.

Let S be a possibly infinite set of pairwise edge-disjoint paths in a graph G. We say that S is *liftable* if no end-vertex of path in S is an internal vertex of another path in S. The operation of *lifting* S consists of deleting all internal vertices of all paths in S, and adding edges joining every pair of non-adjacent vertices of G that are end-vertices of the same path in S. It is easy to see that a graph H is immersed in G if and only if H is isomorphic to a graph obtained from G by deleting a set V of vertices, deleting a set E of edges, and then lifting a liftable set S of paths. Furthermore, a self-immersion of G is proper if and only if at least one of the sets V, E, and S is nonempty.

Given a graph G, a *blob* is a maximal 2-edge-connected subgraph of G. Note that if a graph is 2-edge-connected, the graph itself is also a blob. An easy lemma about the immersion relation can be stated as follows.

Lemma 1. *Let $H \leq_{\mathrm{im}} G$ via the immersion φ and let C be a blob of H. Then there is a blob D of G such that $C \leq_{\mathrm{im}} D$ via the immersion $\varphi|_C$.*

A pair $(\mathscr{G}, \leq)$, where $\mathscr{G}$ is a class of graphs and $\leq$ is a binary relation on $\mathscr{G}$, is called a *quasi-order* if the relation $\leq$ is both reflexive and transitive. A quasi-order $(\mathscr{G}, \leq)$ is a *well-quasi-order* if it admits no infinite antichains and no infinite descending chains.

Date: August 24, 2015.
2010 *Mathematics Subject Classification.* 05C63.
Key words and phrases. graph, immersion, intertwine.

Suppose $(\mathscr{G}, \leq)$ is a quasi-order and G_1 and G_2 are two elements of $\mathscr{G}$. An *intertwine* of G_1 and G_2 is an element G of $\mathscr{G}$ satisfying the following conditions:

- $G_1 \leq G$ and $G_2 \leq G$, and
- if $G' \leq G$ and $G \not\leq G'$, then $G_1 \not\leq G$ or $G_2 \not\leq G$.

The class of all intertwines of G_1 and G_2 is denoted by $\mathscr{I}_{\leq}(G_1, G_2)$. A quasi-order $(\mathscr{G}, \leq)$ satisfies the *finite intertwine property* if for every pair G_1 and G_2 of elements of $\mathscr{G}$, the class of intertwines $\mathscr{I}_{\leq}(G_1, G_2)$ has no infinite antichains. It is clear that if $(\mathscr{G}, \leq)$ is a well-quasi-order, then it also satisfies the finite intertwine property. However, it is well known that the converse is not true; for example, see [4].

Nash-Williams conjectured, and Robertson and Seymour later proved [5] that $(\mathscr{G}_{<\infty}, \leq_{\mathrm{im}})$ is a well-quasi-order, and so it follows that $(\mathscr{G}_{<\infty}, \leq_{\mathrm{im}})$ satisfies the finite intertwine property. In [4], the second author showed that $(\mathscr{G}_{\infty}, \leq_{\mathrm{m}})$, where $\leq_{\mathrm{m}}$ denotes the minor relation, does not satisfy the finite intertwine property. Andreae showed [1] that $(\mathscr{G}_{\infty}, \leq_{\mathrm{im}})$ is not a well-quasi-order. In a result analogous to [4], we strengthen Andreae's result by showing that $(\mathscr{G}_{\infty}, \leq_{\mathrm{im}})$ does not satisfy the finite intertwine property. In particular, we construct two graphs G_1 and G_2, and an infinite class $\mathscr{F}$ in $\mathscr{G}_{\infty}$ such that:

(IT1) $\mathscr{F}$ is an immersion antichain;
(IT2) every graph in $\mathscr{F}$ is connected;
(IT3) both G_1 and G_2 are subgraphs of each graph in $\mathscr{F}$;
(IT4) if G' is properly immersed in a graph G in $\mathscr{F}$, then $G_1 \not\leq_{\mathrm{im}} G'$ or $G_2 \not\leq_{\mathrm{im}} G'$.

Note that (IT3) implies that G_1 and G_2 are immersed in G. Hence, the existence of graphs G_1, G_2 and a class of graphs $\mathscr{F}$ satisfying (IT1)–(IT4) implies the following statement, which is the main result of the paper.

Theorem 2. *The quasi-order $(\mathscr{G}_{\infty}, \leq_{\mathrm{im}})$ does not satisfy the finite intertwine property.*

2. The Construction

We will exhibit two graphs G_1 and G_2 in $\mathscr{G}_{\infty}$ such that $\mathscr{I}_{\leq_{\mathrm{im}}}(G_1, G_2)$ is infinite. The construction of G_1 and G_2 begins with the following results, which are immediate consequences of, respectively, Lemmas 3 and 4, and Lemmas 1 and 2 of [2].

Theorem 3. *There is an infinite set $\mathscr{H}$ of pairwise-disjoint infinite blobs such that $|H| \leq |\mathscr{H}|$ for all $H \in \mathscr{H}$, and $\mathscr{H}$ forms an immersion antichain.*

Theorem 4. *Given an immersion antichain $\mathscr{H}$ of pairwise-disjoint infinite blobs such that $|H| \leq |\mathscr{H}|$ for all $H \in \mathscr{H}$, there is a connected graph G such that the set of blobs of G is $\mathscr{H}$ and G admits no self-immersion except for the trivial one.*

Let $\mathscr{H}$ be an antichain as described in Theorem 3. Partition $\mathscr{H}$ into countably many sets $\{\mathscr{H}_i\}_{i\in\mathbb{Z}}$ with the cardinality of each $\mathscr{H}_i$ equal to $|\mathscr{H}|$. Then, by Theorem 4, for each $i \in \mathbb{Z}$, there is a connected graph B_i whose set of blobs is $\mathscr{H}_i$, and that admits no proper self-immersion. Furthermore, Lemma 1 implies that if i and j are distinct integers, then $B_i \not\leq_{\mathrm{im}} B_j$, as no blob of B_i is immersed in a blob of B_j. Therefore, the set of graphs $\{B_i\}_{i\in\mathbb{Z}}$ is an immersion antichain.

For each graph B_i, label one vertex u_i. Let P be a two-way infinite path with vertices labeled $\{v_i\}_{i\in\mathbb{Z}}$ such that, for each integer i, the vertex v_i is adjacent to v_{i+1} and v_{i-1}. We construct the graph G_1 by taking the disjoint union of P and the graphs B_i for which i is odd, and then identifying the vertices u_i and v_j for $i = j$. Similarly, we construct the graph G_2 by taking the disjoint union of P and the graphs B_i for which i is even, and then identifying the vertices u_i and v_j for $i = j$.

Now let j be an integer. Take the disjoint union of G_1 and all the graphs B_i for which i is even. Then, for each even integer i, identify the vertex v_i of G_1 with the vertex u_{i+2j} of the graph B_{i+2j}. Let F_j be the resulting graph (see Figure 1) and define $\mathscr{F}$ as the set $\{F_j\}_{j\in\mathbb{Z}}$.

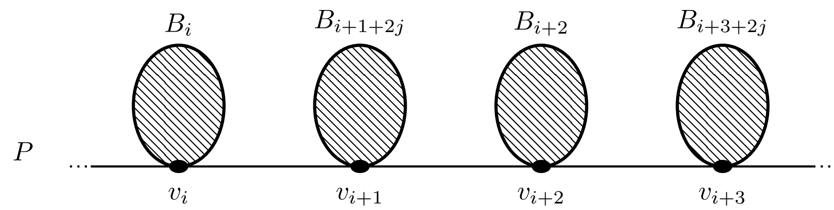

FIGURE 1. The graph F_j

The following lemma immediately implies our main result, Theorem 2.

Lemma 5. *The set of graphs $\mathscr{F} = \{F_j\}_{j\in\mathbb{Z}}$ is an immersion antichain. Furthermore, each $F_j \in \mathscr{F}$ is an immersion intertwine of the graphs G_1 and G_2.*

Proof. Let j be an integer. It is easy to see that F_j satisfies (IT2) and (IT3). Therefore, in order to show that F_j is an immersion intertwine of G_1 and G_2, it suffices to prove that it also satisfies (IT4).

Suppose, for contradiction, that F'_j is a graph that is properly immersed in F_j via a map φ, and both G_1 and G_2 are immersed in F'_j. Then we can obtain F'_j from F_j by deleting a set of vertices V, deleting a set of edges E, and then lifting a liftable set of paths S, with at least one of these sets being nonempty. We consider two cases depending on whether there is an integer i for which B_i meets $V \cup E \cup S$.

First, assume that no B_i meets $V \cup E \cup S$. Then the sets V and S are empty, as all the vertices of F_j are contained in the subgraphs $\{B_n\}_{n\in\mathbb{Z}}$, and E consists of some edges of P.

Suppose the edge $e = \{v_k, v_{k+1}\}$ is in E where k is odd; the argument is symmetric when k is even. The graph $F_j \setminus e$ has exactly two components, with the subgraphs B_k and B_{k+2} in distinct components. Label the component containing B_k as C_1 and the component containing B_{k+2} as C_2.

Let A be a blob of B_k. As A and each blob of C_2 are members of the antichain $\mathscr{A}$, by Lemma 1, we have $A \not\leq_{\mathrm{im}} C_2$. Hence, by transitivity, $B_k \not\leq_{\mathrm{im}} C_2$. It follows similarly that $B_{k+2} \not\leq_{\mathrm{im}} C_1$. But as G_1 is connected and the only components of $F_j \setminus e$ are C_1 and C_2, we have that $G_1 \not\leq_{\mathrm{im}} F_j \setminus e$. Furthermore, as $F'_j \leq_{\mathrm{im}} F_j \setminus e$, by transitivity, $G_1 \not\leq_{\mathrm{im}} F'_j$; a contradiction.

Now suppose that, for some odd integer i, the graph B_i meets $V \cup E \cup S$; again, the argument is symmetric if i is even. As G_1 is immersed in F'_j, so is B_i. Let T be the subgraph of F'_j induced by $\varphi^{-1}(V(B_i) \cup \mathscr{P}(B_i))$, and let ψ be the immersion of B_i into F'_j. As B_i admits no proper self-immersion, there must be some vertex v of B_i such that $\psi(v)$ is a vertex of $F'_j - T$.

Let A_v be the blob of B_i containing v. By Lemma 1, the blob A_v is immersed in some blob of $F'_j - T$. But, again by Lemma 1, each blob of $F'_j - T$ is immersed in a graph of the antichain $\mathscr{A} \setminus \{A_v\}$. So A_v cannot be immersed in $F'_j - T$. Therefore, B_i is not immersed in F'_j and neither is G_1.

Hence, $\mathscr{F}$ satisfies the condition (IT4).

To show that $\mathscr{F}$ is an antichain in $(\mathscr{G}_\infty, \leq_{\mathrm{im}})$, suppose that F_i is immersed in F_j for some distinct integers i and j. By construction, F_i and F_j are not isomorphic. Therefore, F_i is properly immersed in the intertwine F_j and so either $G_1 \not\leq_{\mathrm{im}} F_i$ or $G_2 \not\leq_{\mathrm{im}} F_i$. But both G_1 and G_2 are immersed in F_i by construction; a contradiction. The conclusion follows. $\square$

The graphs $\{B_i\}_{i\in\mathbb{Z}}$ used in our construction, whose existence was proved in [1], have vertex sets of very large cardinality. In fact, the cardinal in question is the first limit cardinal greater than the cardinality of the continuum. It is not known whether the class of graphs of smaller cardinality ordered by the strong immersion relation is a well-quasi-ordering, whether it has the finite intertwine property, and whether there exists a infinite graph of smaller cardinality that admits only the trivial self-immersion.

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